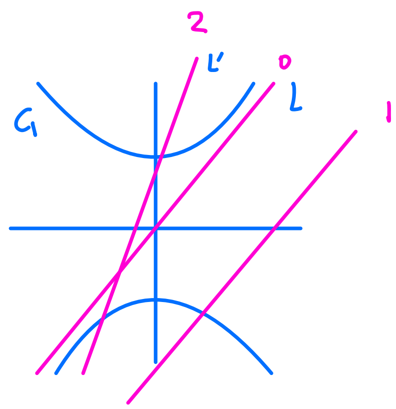
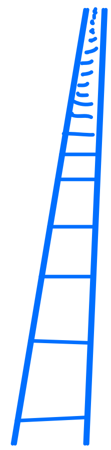
Chapter 4 Projective Varieties

§4.1 projective space

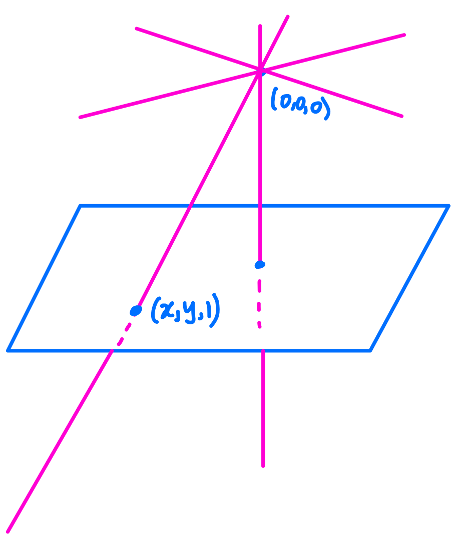




aim: study intersection of two curves.

enlarge the plane s.t. C intersect L at infinity.

from $\mathbb{A}^2$ to $\mathbb{P}^2$:



$$(x,y)\in\mathbb{A}^2 \overset{1:1}{\longleftrightarrow} (x,y,1)\in\mathbb{A}^3 \hookrightarrow \overset{\mathbb{P}^2}{\overset{\parallel}{\{\text{lines through } (0,0,0)\}}}$$

$$(x,y,1) \longmapsto (\ell : \ell \text{ through } (0,0,0) \,\&\, (x,y,1)) \quad ①$$

$\mathbb{P}^n := \mathbb{P}^n(k) := \{ \ell : \text{line in } \mathbb{A}^{n+1} \mid \ell \text{ through } (0,0,\cdots,0) \}$

Projective n-space over k.

$$\mathbb{P}^n = \left(\mathbb{A}^{n+1} \setminus \{(0,\cdots,0)\} \right) / \sim$$

- $P = [x_1 : x_2 : \cdots : x_{n+1}] \in \mathbb{P}^n$

  point in $\mathbb{P}^n$ — homogeneous coordinates

- x_i / x_j is well-defined.

- $U_i := \{ [x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n \mid x_i \neq 0 \} \cong \mathbb{A}^n$

- $\mathbb{P}^n = \bigcup_{i=1}^{n+1} U_i$

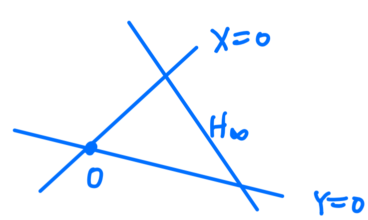
$$H_\infty := \mathbb{P}^n \setminus U_{n+1} = \{ [x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n \mid x_{n+1} = 0 \} \cong \mathbb{P}^{n-1}$$

hyperplane at infinity.

Example: 1) $\mathbb{P}^1 \cong \mathbb{A}^1 \cup \{\infty\}$. $U_1 \cong \mathbb{A}^1$, $U_2 \cong (\mathbb{A}^1 \setminus \{0\}) \cup \{\infty\}$.

2) $\mathbb{P}^2 \cong \mathbb{A}^2 \cup H_\infty$

parallel lines ?



$\mathbb{A}^n$ affine space $\rightsquigarrow$ $\mathbb{P}^n$ projective space

$V(F) = \{P \in \mathbb{A}^n \mid F(P)=0\}$ $\rightsquigarrow$??

$P = [x_1 : \cdots : x_{n+1}]$ stands for the line through 0 & $(x_1, \cdots, x_{n+1})$.

$P = [x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n$ is called a zero of $F \in k[X_1, \cdots, X_{n+1}]$ if

$$F(\lambda x_1, \lambda x_2, \cdots, \lambda x_{n+1}) = 0 \quad \forall \lambda \in k.$$

We simply write $F(P)=0$.

$\forall S \subseteq k[X_1, \cdots, X_{n+1}]$. projective algebraic set.

$$V(S) := \{P \in \mathbb{P}^n \mid F(P)=0, \ \forall F \in S\}$$

$\forall X \subseteq \mathbb{P}^n$ the ideal of X.

$$I(X) := \{F \in k[x_1, \cdots, x_{n+1}] \mid F(P) = 0 \quad \forall P \in X\}$$

Fact: $F = F_r + F_{r+1} + \cdots + F_d$ (F_i = form of deg i). Then

$$F(P) = 0 \Leftrightarrow F_i(P) = 0 \quad \forall i = r, \cdots, d.$$

$I \lhd k[X_1, \cdots, X_{n+1}]$ is called homogeneous if

$$\forall F = F_0 + \cdots + F_m \in I \Rightarrow F_i \in I \quad \forall i = 0, \cdots, m.$$

i.e. $I = I_0 \oplus I_1 \oplus \cdots$

Fact: 1): $V(S) = V(\langle S \rangle) = V(\text{min. homog. ideal containing } S)$

$\langle S \rangle \triangleleft k[x_1, \dots, x_{n+1}]$ (the ideal generated by S)

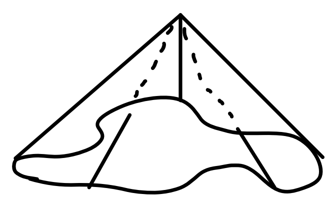
2) $I(X)$ is homogeneous

$$\{\text{homog. ideals in } k[x_1, \dots, x_{n+1}]\} \underset{I}{\overset{V}{\rightleftarrows}} \{\text{alg. sets in } \mathbb{P}^n\}$$

- $V = \text{irr.} \Leftrightarrow I(V) = \text{prime}$
- irr. decomposition.

projective variety := **irreducible** algebraic set in $\mathbb{P}^n$.

$$\begin{tikzcd}
\mathbb{A}^{n+1} \setminus (0, \dots, 0) \arrow[r, two heads, "\pi"] & \mathbb{P}^n \\
\pi^{-1}(V) \arrow[u, phantom, "\cup" rotate=90] \arrow[r, two heads] & V \arrow[u, phantom, "\cup" rotate=90]
\end{tikzcd}$$



$C(V) := \pi^{-1}(V) \cup \{(0, \dots, 0)\}$

↑ cone over V

Fact: 1) $V \neq \phi$, then $I_a(C(V)) = I_p(V)$

2) $I \triangleleft k[x_1, \dots, x_{n+1}]$ homog. $V_p(I) \neq \phi$, then

$$C(V_p(I)) = V_a(I)$$